(12) LEVEL II

TECHNICAL REPORT T-79-49

U.S. ARMY MISSILE RESEARCH AND DEVELOPMENT COMMAND

SAM: AN ALTERNATIVE TO SAMPLED-DATA SIGNAL FLOW GRAPHS

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER M-T-79-49	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) SAM: AN ALTERNATIVE TO SAMPLED-DATA SIGNAL FLOW GRAPHS		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) S.M. Seltzer		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Commander US Army Missile Research and Development Command ATTN: DRDMI-TG Redstone Arsenal, Alabama 35809		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Commander US Army Missile Research and Development Command ATTN: DRDMI-TI Redstone Arsenal, Alabama 35809		12. REPORT DATE 10 May 1979
		13. NUMBER OF PAGES 38
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Sampled-Data   SAM
Digital
Guidance and Control
Signal Flow Graphs

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report describes an alternative to the use of Signal Flow Graphs and Mason's Gain Rule for the analysis of complicated digital control systems. The technique is analytical in nature and makes use of a systematic manipulation of algebraic equations describing the system to be analyzed. The technique is modified to be amenable to a new signal flow graph method for the analyst who prefers using signal flow graphs. Both the analytical technique and the modified signal flow graph technique are applied to examples and compared to a standard signal flow graph technique to display their

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relative advantages.

CONTENTS

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ILLUSTRATIONS

1. INTRODUCTION

This report describes an alternative to the use of Signal Flow Graphs (SFG) and Mason's Gain Rule (Formula) for analysis of complicated sampled-data control systems. Usually in such systems, block diagram algebraic manipulation may become unwieldly, particularly when such systems include multiple loops and samplers. The Systematic Analysis Method (SAM) may be applied to such systems, as well as to simple single-loop feedback systems. This is shown in Section 2. Also shown is how to apply SAM to make use of modified z-transforms (Section 5).

The advantages of using SAM are that the cumbersome application of Mason's Gain Formula can be avoided. Further, the entire method of drawing Signal Flow Graphs may be circumvented. Since only the equations describing the system are needed for SAM, even the customary block diagram is not needed.

If the analyst prefers to use one of the Signal Flow Graph methods, a modified SFG technique is also described (Section 3). It is simpler and less cumbersome to apply than the conventional SFG method, which for purposes of comparison is described in Section 4.

All three techniques are applied to two examples. This is done to better describe the application of SAM and the modified SFG and to help provide a basis for comparison (Section 6) of the three methods.

To obviate searching for such a description, Mason's Gain Rule is described in the appendix.

2. SYSTEMATIC ANALYTICAL METHOD (SAM)

SAM is implemented by performing the following four steps. If the equations resulting from the first three steps are placed in a table of three columns (one for each step), they are easily manipulated to perform the fourth and final step.

STEP NO. 1.   OBTAIN "SYSTEM EQUATIONS"

The equations describing the system are written in the Laplace domain. If the system is described by block diagram, the "system equations" are written upon inspection.

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STEP NO. 2.   OBTAIN "MODIFIED SYSTEM EQUATIONS"

If any of the "system equations" contain terms that in themselves contain <u>the product(s) of an unsampled system variable and an unsampled transfer function</u>, then the unsampled variable must be replaced by an expression containing no unsampled variable(s). In complex systems, this may require a chain of several substitutions.

An "unsampled variable" is recognized as one upon which the pulse transform operation has not taken place. For example, when the pulse transform of a Laplace transform function and/or variable is taken, that operation is denoted symbolically by placing an asterisk immediately following the expression, yielding a so-called "starred quantity." For example, the pulse transform of the Laplace function F(s) is denoted as F*(s). One manner of expressing F*(s) in terms of F(s) is

$$F*(s) = \frac{1}{T} \sum_{n=-\infty}^{\infty} F\ (s + i\ 2\Pi\ n/T) \qquad (1)$$

where T represents the sampling period.

STEP NO. 3.   OBTAIN "PULSED SYSTEM EQUATIONS"

Pulse transforms are now taken off each side of the "modified system equations," yielding "pulsed system equations." Now all system variables, either unsampled or sampled (pulsed), may be solved for, either in the "system equations" or the "pulsed system equations."

STEP NO. 4.   OBTAIN DESIRED INPUT/OUTPUT RELATIONSHIPS

All system variables, both starred and unstarred, may be found in either the "system equations" (Step No. 1) or the "pulsed system equations" (Step No. 3). The desired output(s) may be solved for by selecting the appropriate "system" or "pulsed system" equations, substituting as necessary. This will be brought forth in the examples.

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EXAMPLE NO. 1

Given: The digital system of Example No. 1 is described by the block diagram of <u>Figure 1</u>.

To Find: The continuous-data and pulsed (sampled) outputs, C(s) and C*(s), respectively, in terms of the system input R(s) and the system transfer functions G(s) and H(s).

STEP 1. SYSTEM EQUATIONS[1]

Obtain these equations directly upon inspection of <u>Figure 1</u>.

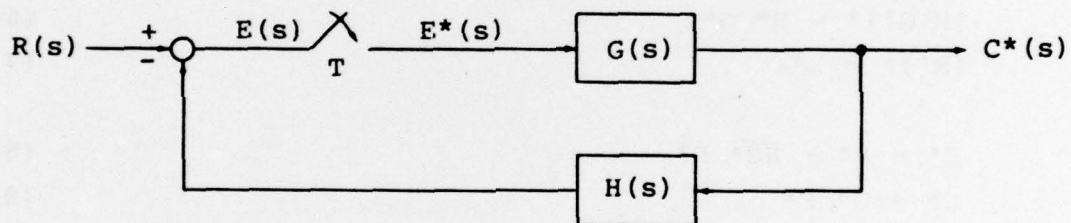$$E = R - HC \tag{2}$$
$$C = G E^* \tag{3}$$

Figure 1. Block Diagram for Example No. 1.

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1. <u>Note</u>: In the sequel, the following shorthand notation will be used:

$F \overset{d}{=} F(s) = \mathcal{L}\{f(t)\}$, i.e. the Laplace transform of F(t),

$F^* \overset{d}{=} F^*(S)$,

$\overline{GH}^* \overset{d}{=} [G(s)H(s)]^*$.

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STEP 2. MODIFIED SYSTEM EQUATIONS

One sees that Equation (2) contains a product of an unsampled (i.e., unstarred) system variable, C, and an unsampled transfer function, H. Since this violates a condition stated in the description of Step No. 2, a substitute must be found for C. This is obvious in Equation (3), which is substituted into Equation (2) to yield the acceptable form,

$$E = R - HG\ E^*, \tag{4}$$

i.e., it contains no products of unsampled variables and unsampled transfer functions. The product of unsampled transfer functions, HG [i.e., H(s) G(s)], is acceptable.

STEP NO. 3. PULSED SYSTEM EQUATIONS

Take the pulse transform of each side of each of the "modified system equations," making use of the following rules:

$$[RG]^* = \overline{RG}^*, \tag{5}$$

$$[R\ G^*]^* = R^*\ G^*, \tag{6}$$

$$[R^*]^* = R^*. \tag{7}$$

$$E^* = R^* - \overline{HG}^*\ E^* \tag{8}$$

$$C^* = G^*\ E^* \tag{9}$$

The resulting equations from Step Nos. 1, 2 and 3 can be placed in a table (Table 1), while they are being developed, for systematic orderliness.

TABLE 1. SYSTEM EQUATIONS FOR EXAMPLE NO. 1

Sys. Eqs.		Mod. Sys. Eqs.		Pulsed Sys. Eqs.	
E = R - HC	(2)	E = R - HG E*	(4)	$E^* = R^* - \overline{HG}^*\ E^*$	(8)
C = G E*	(3)	C = G E*	(3)	$C^* = G^*\ E^*$	(9)

STEP 4. OBTAIN, C*, C

First solve Equation (8) for E* :

$$E^* = \left[\frac{1}{(1 + \overline{HG}^*)}\right] R^*. \tag{10}$$

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Substituting this expression for E* into Equation (9), one obtains

$$C* = \left[\frac{G}{(1 + \overline{HG}*)} \right] R*.$$ (11)

"to obtain C, one substitutes Equation (10) into Equation (3), yielding"

$$C = \left[\frac{G}{(1 + HG*)} \right] R*.$$ (12)

EXAMPLE NO. 2.

Given: The digital system described by the block diagram of Figure 2.

To Find: The continuous-data and pulsed (sampled) outputs, C(s) and C*(s), respectively, in terms of the system input R(s) and the system transfer functions.
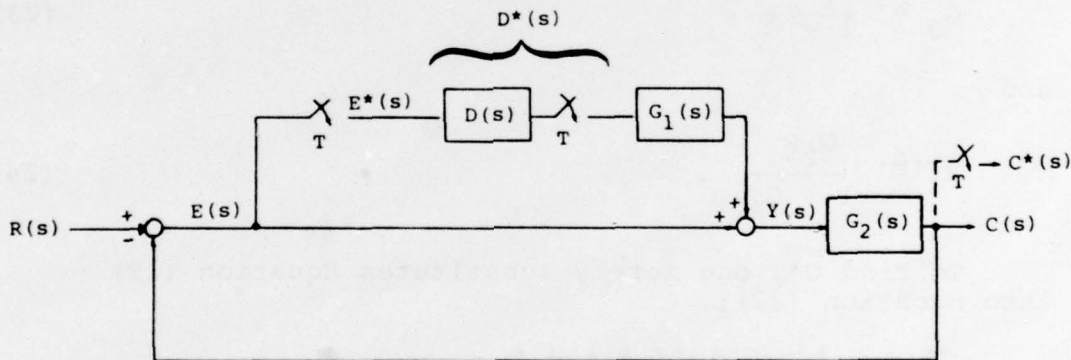
Figure 2. Block Diagram for Example No. 2.

In the interest of being systematic, the analyst may wish to assign "states" (X_j) to the various system variables. The "system equations" (Step No. 1) are written from inspection of Figure 2. In order to implement Step No. 2, the "system equations" of Step No. 1 must be checked for possible products of unsampled system variables and unsampled transfer functions. Two such products exist: G_1X_2 in Equation (15) and G_2X_3 in Equation (16). The first product is readily manipulated into

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the approved form by making use of the relation between X_2 and X^*_1 of Equation (14), which is substituted into Equation (15) to yield Equation (17). The product, G_2X_3, is easily handled by substituting Equation (17) into Equation (16), then substituting Equation (13) for X_1, and solving the resulting expression for X_4, yielding Equation (18). This completes Step No. 2 (the second column of the array of Table 2). The third column (Step No. 3) is obtained merely by "starring" each side of each of the equations in the second column.

TABLE 2.   SYSTEM EQUATIONS FOR EXAMPLE NO. 2

Sys. Eqs.		Mod. Sys. Eqs.		Pulsed Sys. Eqs.	
$E \equiv X_1 = R - X_4$	(13)	$X_1 = R - X_4$	(13)	$E^* = X^*_1 = R^* - X_4^*$	(19)
$E^* \equiv X_2 = X_1^*$	(14)	$X_2 = X_1^*$	(14)	$E^* = X_2^* = X_1^*$	(20)
$Y \equiv X_3 = D^*G_1X_2 + X_1$	(15)	$X_3 = D^*G_1X_1^* + X_1$	(17)	$Y^* = X_3^* = D^*G_1^*X_1^* + X_1^*$	(21)
$C \equiv X_4 = G_2X_3$	(16)	$X_4 = \dfrac{G_2(D^*G_1X_1^* + R)}{1 + G_2}$	(18)	$C^* = X_4^* = G_3^*D^*X_1^* + R_1^*$	(22)

where

$$G_3 \overset{\underline{d}}{=} \frac{G_1G_2}{1 + G_2} \tag{23}$$

and

$$R_1 \overset{\underline{d}}{=} \frac{G_2R}{1 + G_2} \ . \tag{24}$$

To find C^*, one merely substitutes Equation (19) into Equation (22):

$$C^* = X_4^* = G_3^*D^*X_1^* + R_1^*$$

$$= G_3^*D^*(R^* - X_4^*) + R_1^*$$

$$= \frac{G_3^*D^*R^* + R_1^*}{1 + G_3^*D^*} \ . \tag{25}$$

In a similar manner, one finds C by substituting Equation (15) into Equation (16), substituting Equations

(14) and (13) for X_2 and X_1, respectively, and finally substituting Equation (19) for the resulting X_1*, i.e.

$$C = X_4 = G_2 X_3$$

$$= G_2(D*G_1 X_2 + X_1)$$

$$= G_2(D*G_1 X_1* + R - X_4)$$

$$= \frac{G_1 G_2 D* X_1* + G_2 R}{1 + G_2}$$

$$= G_3 D* X_1* + R_1$$

$$= G_3 D*(R* - X_4*) + R_1. \tag{26}$$

Since $X_4* = C*$ was just determined in Equation (25), that value is substituted into Equation (26) to yield:

$$C = G_3 D* \left[R* - \left(\frac{G_3* D* R* + R_1*}{1 + G_3* D*} \right) \right] + R_1$$

$$= G_3 D* \left(\frac{R* + G_3* D* R* - G_3* D* R* - R_1*}{1 + G_3* D*} \right) + R_1$$

$$= \left(\frac{G_3 D*}{1 + G_3* D*} \right) (R* - R_1*) + R_1 \quad . \tag{27}$$

An alternate form for $(R* - R_1*)$ may be found if desired:

$$R* - R_1* = R* - \left(\frac{G_2 R}{1 + G_2} \right)*$$

$$= \left[\left(\frac{1 + G_2}{1 + G_2} \right) R \right]* - \left(\frac{G_2 R}{1 + G_2} \right)*$$

$$= \left(\frac{R}{1 + G_2} \right)* + \left(\frac{G_2 R}{1 + G_2} \right)* - \left(\frac{G_2 R}{1 + G_2} \right)*$$

$$= \left(\frac{R}{1 + G_2} \right)* \stackrel{d}{=} R_2*. \tag{28}$$

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This expression for $(R^* - R_1^*)$ may be substituted into Equation (27) to yield

$$C = \frac{G_3 D^*}{1 + G_3^* D^*} \, R_2^* + R_1, \qquad (29)$$

which may be slightly simpler in form than Equation (27).

3. MODIFIED SIGNAL FLOW GRAPH TECHNIQUE

If the analyst prefers to use a Signal Flow Graph (SFG) technique, the following modified SFG is proposed. It incorporates many of the features developed in SAM. As such, it appears to be simpler to implement, requiring the application of Mason's Gain Rule at only one stage of the analysis.

The first three steps are identical to those of SAM.

STEP NO. 4.   CONSTRUCT EQUIVALENT SFG

The equivalent SFG is drawn directly from the information contained in the "system equations" (Step No. 1) or from the block diagram.

STEP NO. 5.   CONSTRUCT SAMPLED SFG

The sampled SFG is drawn from the "pulsed system equations" (Step No. 3).

STEP NO. 6.   CONSTRUCT COMPOSITE SFG

This is achieved by connecting the output nodes of the samplers in the equivalent SFG to the nodes representing those same quantities on the sampled SFG.

STEP NO. 7.   OBTAIN DESIRED INPUT/OUPUT RELATIONSHIPS

Mason's Gain Rule (Appendix A) is applied to the composite SFG to obtain desired outputs in terms of system transfer functions and inputs to the system.

Examples of the use of the modified SFG method follow. For comparative purposes, the same two examples to which SAM was applied will be used.

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EXAMPLE NO. 1.

STEP NOS. 1 - 3. SYSTEM, MODIFIED SYSTEM, AND PULSED SYSTEM EQUATIONS

Repeat these steps as shown in Example No. 1 of Section 2 (SAM). They are summarized in Table 1 and are Equations (2), (3); (4), (3); and (8), (9), respectively.

STEP NO. 4. CONSTRUCT EQUIVALENT SFG

The equivalent SFG is constructed directly from the "system equations" of Step No. 1, Equations (2) and (3). The resulting SFG is shown as Figure 3.

Figure 3. Equivalent SFG for Example No. 1.

STEP NO. 5. CONSTRUCT SAMPLED SFG

The sampled SFG is constructed directly from the "pulsed system equations" of Step No. 3, Equations (8) and (9). The resulting SFG is shown as Figure 4.

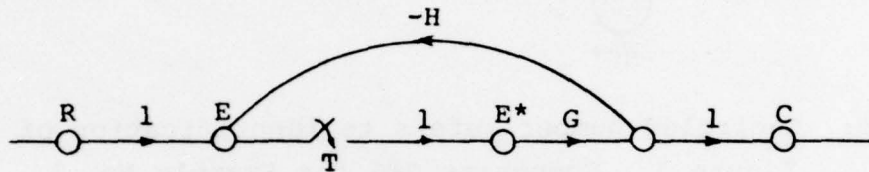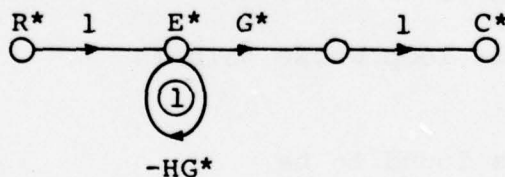Note: Encircled number refers to identification of loop.

Figure 4. Sampled SFG for Example No. 1.

STEP NO. 6. CONSTRUCT COMPOSITE SFG

The composite SFG is constructed by joining the two SFG's of Figures 4 and 5 in the prescribed manner. In

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this example a single line connecting the E*'s of the two SFG's is required (Figure 5).
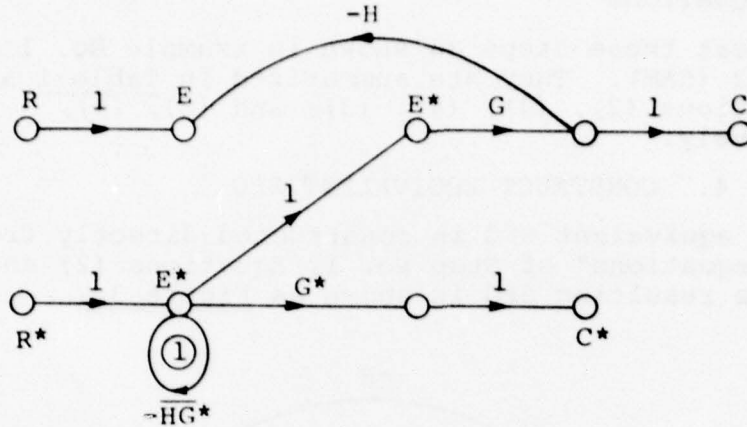
Note: Encircled number refers to identification of loop.

Figure 5. Composite SFG for Example No. 1.

STEP NO. 7. OBTAIN C*, C

Looking at Figure 5, one sees that there are two inputs to the system, R and R*. Applying Mason's Gain Rule (appendix), one finds only one possible forward path from R* to C* and none from R; hence, k = 1. The gain along that forward path, M_k, is seen to be

$$M_k = M_1 = G* .$$ (30)

There is a single loop whose gain is

$$K_1 = - \overline{HG}*.$$ (31)

The value of Δ is found to be

$$\Delta = 1 - K_1 = 1 + HG*.$$ (32)

Since the single forward path touches the single loop of this system,

$$\Delta_k = \Delta_1 = 1.$$ (33)

The gain, M, between C* and the input R* is then

$$M = \frac{C*}{R*} = \frac{M_1 \Delta_1}{\Delta} = \frac{G*}{1 + \overline{HG}}*.$$ (34)

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Solving Equation (34) for C* we obtain the same result as Equation (11).

In solving for C, we find that only input R* has a forward path to C. In this case the gain along that path is seen to be G (Figure 5), i.e.

$$M_1 = G. \tag{35}$$

The value of Δ remains the same as that shown in Equation (32), as does the value of K_1 remain as shown in Equation (31). The single forward path touches the single loop of the system, so Equation (33) still applies. The gain, M, between C and the input R* is then

$$M = \frac{C}{R^*} = \frac{M_1 \Delta_1}{\Delta} = \frac{G}{1 + \widetilde{HG}}^*. \tag{36}$$

Solving Equation (36) for C one obtains the same result as Equation (12).

EXAMPLE NO. 2

STEP NOS. 1 - 3

Repeat these steps as shown in Example No. 2 of Section 2 (SAM). They are summarized in Table 2.

STEP NO. 4. EQUIVALENT SFG

Construct the equation SFG directly from "system equations"(13) through (16) (Figure 6).
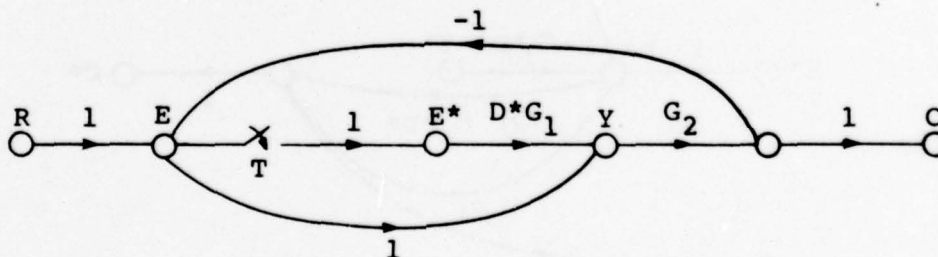
Figure 6. Equivalent SFG for Example No. 2.

STEP NO. 5. SAMPLED SFG

Construct the sampled SFG directly from "pulsed system equations" (19) - (22) (Figure 7).

Figure 7. Sampled SFG for Example No. 2 (modified SFG method).

STEP NO. 6. COMPOSITE SFG
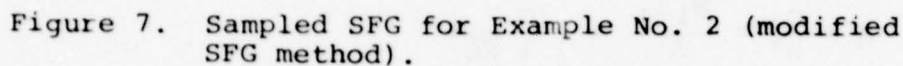
The composite SFG is obtained by joining the two SFG's of Figures 6 and 7 with a line connecting the E*'s (X_2's) of the two SFG's (Figure 8).

Note: Encircled numbers refer to identification of loops.
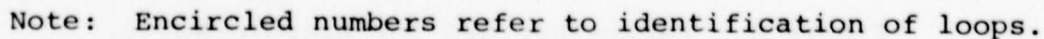
Figure 8. Composite SFG for Example No. 2 (modified SFG method).

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STEP NO. 7.   OBTAIN C*, C

From Figure 8 one sees that there are three inputs to the system: R, R*, and R_1*. Applying Mason's Gain Rule, one finds two forward paths to C*, one from R* and one from R_1*; hence, k = 2.

There are two loop gains, denoted herein as K_1 and K_2. The loops are designated by encircled numbers on Figure 8, and their gains are

$$K_1 = -G_2,$$ (37)

$$K_2 = -G_3^* D^*.$$ (38)

It is observed that the loops are nontouching (necessary information for formulating Δ). Δ is thus found to be

$$\Delta = 1 - (K_1 + K_2) + K_1 K_2$$

$$= 1 - (-G_2 -G_3^* D^*) + G_2 G_3^* D^*$$

$$= (1 + G_2) (1 + G_3^* D^*).$$ (39)

The forward path from R* to C* may be designated as k = 1. Since it is touched by Loop 2 but not by Loop 1, the value of Δ_1 is

$$\Delta_1 = 1 - K_1$$

$$= 1 + G_2.$$ (40)

The forward path from R_1* to C* may be designated as k = 2. Since it is touched by Loop 2 but not by Loop 1,

$$\Delta_2 = \Delta_1.$$ (41)

The gain along the first forward path (k = 1) is, from Figure 8,

$$M_1 = G_3^* D^*.$$ (42)

The gain along the second forward path (k = 2) is

$$M_2 = 1.$$ (43)

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The gain M_1 between R^* and C^* is

$$M^1 \stackrel{d}{=} \cdot \frac{C^{1*}}{R^*} = \frac{M_1 \Delta_1}{\Delta}$$

$$= \frac{(G_3^*D^*)(1 + G_2)}{(1 + G_2)(1 + G_3^*D^*)}$$

$$= \frac{G_3^*D^*}{1 + G_3^*D^*} \cdot \tag{44}$$

The gain M^2 between R_1^* and C^* is

$$M^2 \stackrel{d}{=} \cdot \frac{C^{2*}}{R_1^*} = \frac{M_2 \Delta_2}{\Delta}$$

$$= \frac{(1)(1 + G_2)}{(1 + G_2)(1 + G_3^*D^*}$$

$$= \frac{1}{1 + G_3^*D^*} \cdot \tag{45}$$

Solving Equations (44) and (45) each for C^{1*} and C^{2*}, respectively,

where

$$C^* = C^{1*} + C^{2*}, \tag{46}$$

one finally obtains

$$C^* = M^1 R^* + M^2 R_1^*$$

$$= \left[\frac{G_3^*D^*}{1 + G_3^*D^*}\right] R^* + \left(\frac{1}{1 + G_3^*D^*}\right) R_1^*$$

$$= \frac{G_3^*D^*R^* + R_1^*}{1 + G_3^*D^*} , \tag{47}$$

which is seen to be identical with the earlier SAM result of Equation (25).

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To obtain C, one must first observe from Figure 8 that all three inputs to the system can find their way to the node representing C. The two loop gains, K_1 and K_2, are the same for finding C*, as is Δ. The forward path from input R* to C, designated as $k = 1$, has a gain M_1 of

$$M_1 = D^* G_1 G_2. \tag{48}$$

The gain along the path between R_1^* and C, designated as $k = 2$, has a gain M_2 of

$$M_2 = -D^* G_1 G_2 . \tag{49}$$

Finally, the gain along the path between R and C, designated as $k = 3$, has a gain M_3 of

$$M_3 = G_2 . \tag{50}$$

The $k = 1$ and $k = 2$ paths touch both loops, so the values of Δ_1 and Δ_2 are both unity. The $k = 3$ path only touches Loop 1, so the values of Δ_3 is

$$\Delta_3 = 1 - K_2 = 1 + G_3^* D^*. \tag{51}$$

The gain M^1 between R* and C is

$$M^1 \overset{\text{d}}{=} \frac{C^1}{R^*} = \frac{M_1 \Delta_1}{\Delta}$$

$$= \frac{D^* G_1 G_2}{(1 + G_2)(1 + G_3^* D^*)} . \tag{52}$$

The gain M^2 between R_1^* and C is

$$M^2 \overset{\text{d}}{=} \frac{C^2}{R_1^*} = \frac{M_2 \Delta_2}{\Delta}$$

$$= \frac{-D^* G_1 G_2}{(1 + G_2)(1 + G_3^* D^*)} . \tag{53}$$

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The gain M^3 between R and C is

$$M^3 = \frac{C^3}{R} = \frac{M_3 \Delta_3}{\Delta}$$

$$= \frac{G_2(1 + G_3{}^*D^*)}{(1 + G_2)(1 + G_3{}^*D^*)}$$

$$= \frac{G_2}{1 + G_2} . \tag{54}$$

Solving Equations (52) through (54) for C^1, C^2, and C^3, respectively,

where

$$C = C^1 + C^2 + C^3 , \tag{55}$$

one obtains

$$C = M^1 R^* + M^2 R_1{}^* + M^3 R$$

$$= \frac{D^*G_1G_2 R^*}{(1 + G_2)(1 + G_3{}^*D^*)} - \frac{D^*G_1G_2 R_1{}^*}{(1 + G_2)(1 + G_3{}^*D^*)} + \frac{G_2 R}{1 + G_2}$$

$$= \frac{D^*G_3}{(1 + G_3{}^*D^*)} (R^* - R_1{}^*) + R_1 . \tag{56}$$

Equation (28) may be used in an attempt to simplify the above result, yielding

$$C = \frac{D^*G_3}{(1+G_3{}^* D^*)} R_2{}^* + R_1 . \tag{57}$$

This is equivalent to Equation (29) obtained using SAM.

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4. THE SAMPLED SIGNAL FLOW GRAPH METHOD

The standard Signal Flow Graph method in use is the "Sampled Signal Flow Graph Method."[2] So that the SAM and modified SFG methods exposed in Section 2 and 3, respectively, of this report may be compared to this standard method, it is described briefly in this section. The same two examples that have been used previously in this report are used in this section to better permit comparison of the various methods. The other popularly used SFG method, "The Direct Signal Flow Graph Method," will not be described herein.[3]

STEP NO. 1.   CONSTRUCT EQUIVALENT SFG

This is equivalent to Step No. 4 of the modified SFG procedure of Section 3.

STEP NO. 2.   CONSTRUCT SAMPLED SFG

Write system equations for all noninput nodes of SFG, applying Mason's Gain Rule. A "noninput node" is defined as a node that is not an input node, where an "input" is defined as a system input or the output of a sampler.

Take the pulse transform of each side of each of the system equations.

Using equations noted in the paragraph above, draw the sampled SFG for the system.

STEP NO. 3.   OBTAIN RELATION BETWEEN SAMPLED INPUTS/OUTPUTS

This is achieved by applying Mason's Gain Rule to the SFG.

STEP NO. 4.   OBTAIN RELATION BETWEEN INPUTS/
              CONTINUOUS-DATA OUTPUTS

Connect the SFG's of Steps No. 1 and No. 2 to yield a composite SFG.

Apply Mason's Gain Rule to SFG.

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2.   B. C. Kuo, Digital Control Systems, SRL Publishing Company, Champaign, Illinois, 1977, pp. 100-105.

3.   Ibid, pp. 106-115.

<u>EXAMPLE NO. 1.</u>

See Section 2 and <u>Figure 1</u> for a description of the example.

STEP NO. 1.   EQUIVALENT SFG

From the block diagram describing the system (<u>Figure 1</u>), construct an equivalent SFG. This is the same as <u>Figure 3</u>.

STEP NO. 2.   SAMPLED SFG

Write system equations, applying Mason's Gain Rule to equivalent SFG (<u>Figure 3</u>).

$$E = R - G H E^*$$ (58)

$$C = G E^*$$ (59)

Take the pulse transform of each side of the system equations.

$$E^* = R^* - \overline{GH}^* E^*$$ (60)

$$C^* = G^* E^*$$ (61)

Draw sampled SFG from pulsed system equations (<u>Figure 4</u>).

STEP NO. 3.

Obtain C^*, E^* from sampled SFG (<u>Figure 4</u>), applying Mason's Gain Rule. There is one path from R^* to E^*; $k = 1$.

$$M_1 = 1$$ (62)

$$1 \text{ loop: } K_1 = -\overline{GH}^*$$ (63)

$$\Delta = 1 - K_1 = 1 + \overline{GH}^*$$ (64)

The forward path touches the loop:

$$\Delta_1 = 1,$$ (65)

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$$M^1 \stackrel{d}{=} \frac{E^*}{R^*} = \frac{M_1 \Delta_1}{\Delta} = \frac{1}{1 + \overline{GH}^*} \quad . \tag{66}$$

Solving Equation (66) for E^* leads to

$$E^* = \frac{R^*}{1 + \overline{GH}^*} \quad . \tag{67}$$

There is one path from R^* to C^*, $k = 2$.

$$M_2 = G^* \tag{68}$$

$$\Delta_2 = \Delta_1 \tag{69}$$

$$M^2 = \frac{C^*}{R^*} = \frac{M_2 \Delta_2}{\Delta} = \frac{G^*}{1 + \overline{GH}^*} \tag{70}$$

Solving Equation (70) for C^* leads to

$$C^* = \frac{G^*}{1 + \overline{GH}^*} R^* \quad . \tag{71}$$

STEP NO. 4.   OBTAIN, C, E

Connect SFG's of Figures 3 and 4 to form composite SFG (Figure 5).

Apply Mason's Gain Rule to Figure 5.   There is one path from R^* to C, none from R: $k = 1$.

$$M_1 = G \tag{72}$$

$$K_1 = -\overline{GH}^* \tag{73}$$

$$\Delta_1 = 1 \tag{74}$$

$$M^1 \stackrel{d}{=} \frac{C}{R^*} = \frac{M_1 \Delta_1}{\Delta} = \frac{G}{1 + \overline{GH}^*} \tag{75}$$

Solving Equation (75) for C leads to

$$C = \frac{G}{1 + \overline{GH}^*} R^* \quad . \tag{76}$$

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There is one path from R* to E (k = 2), and one
path from R (k = 3).

$$M_2 = -GH \tag{77}$$

$$M_3 = 1 \tag{78}$$

$$\Delta_2 = 1 \tag{79}$$

$$\Delta_3 = 1 - K_1 = 1 + \overline{GH}^* \tag{80}$$

$$M^2 \stackrel{d.}{=} \frac{E^2}{R^*} = \frac{M_2 \Delta_2}{\Delta} = \frac{-GH}{1 + \overline{GH}^*} \tag{81}$$

$$M^3 \stackrel{d.}{=} \frac{E^3}{R} = \frac{M_3 \Delta_3}{\Delta} = \frac{1 + \overline{GH}^*}{1 + \overline{GH}^*} = 1 \tag{82}$$

Solving Equations (81) and (82) for E^2 and E^3, respectively,
and using the relationship,

$$E = E^2 + E^3, \tag{83}$$

one obtains

$$E = R - \frac{GH}{1 + \overline{GH}^*} R^* . \tag{84}$$

In this elementary example, it is seen that the SFG's and
algebraic relationships are identical to those obtained
with the modified SFG method of Section 3. Such will not
be the case with Example 2.

EXAMPLE NO. 2

See Section 2 and Figure 2 for the description of the
digital control system. It is desired to find C and C*
in terms of system input R and the system transfer
functions.

STEP NO. 1. EQUIVALENT SFG

This may be drawn directly from the system equations
summarized in column 1 of Table 2. It is shown as
Figure 6.

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STEP NO. 2.   SAMPLED SFG

Write system equations, applying Mason's Gain Rule to SFG of Figure 6.   There is one path from R to E (k = 1), one path from R to Y (k = 2), and one path from R to C (k = 3).   There is one path from E* to Y (k = 4), one path to C (k = 5), and one path to E.

Using the techniques that are by now well established in this report,

$$M_1 = 1, \tag{85}$$

$$M_2 = 1, \tag{86}$$

$$M_3 = G_2, \tag{87}$$

$$M_4 = D^*G_1, \tag{88}$$

$$M_5 = D^*G_1\,G_2, \tag{89}$$

$$M_6 = -D^*G_1\,G_2, \tag{90}$$

$$K_1 = -G_2, \tag{91}$$

$$\Delta = 1 - K_1 = 1 + G_2, \tag{92}$$

$$\Delta_k = 1, \tag{93}$$

$$M^1 \underset{\cdot}{\underline{\underline{d}}} \frac{E^1}{R} = \frac{M_1\Delta_1}{\Delta} = \frac{1}{1 + G_2}, \tag{94}$$

$$M^2 \underset{\cdot}{\underline{\underline{d}}} \frac{Y^1}{R} = \frac{M_2\Delta_2}{\Delta} = \frac{1}{1 + G_2}, \tag{95}$$

$$M^3 \underset{\cdot}{\underline{\underline{d}}} \frac{C^1}{R} = \frac{M_3\Delta_3}{\Delta} = \frac{G_2}{1 + G_2}, \tag{96}$$

$$M^4 \underset{\cdot}{\underline{\underline{d}}} \frac{Y^2}{E^*} = \frac{M_4\Delta_4}{\Delta} = \frac{D^*G_1}{1 + G_2}, \tag{97}$$

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$$M^5 \stackrel{d.}{=} \cdot \frac{C^2}{E^*} = \frac{M_5 \Delta_5}{\Delta} = \frac{D^* G_1 G_2}{1 + G_2} \ , \tag{98}$$

$$M^6 \stackrel{d.}{=} \cdot \frac{E^2}{E^*} = \frac{M_6 \Delta_6}{\Delta} = \frac{-D^* G_2 G_2}{1 + G_2} \ . \tag{99}$$

One solves Equations (94) and (99) for E^1 and E^2, respectively, and using the expression,

$$E = E^1 + E^2, \tag{100}$$

and one obtains E:

$$E = M^1 R + M^6 E^*$$

$$= \frac{R}{1 + G_2} - \frac{D^* G_1 G_2}{1 + G_2} E^*$$

$$= R_2 - D^* G_3 E^*. \tag{101}$$

One solves Equations (95) and (97) for Y^1 and Y^2, and using the expression,

$$Y = Y^1 + Y^2, \tag{102}$$

one obtains Y:

$$Y = M^2 R + M^4 E^*$$

$$= R_2 + \frac{D^* G_1}{1 + G_2} E^*. \tag{103}$$

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Similarly, one solves Equations (96) and (99) for C^1 and C^2, and using the expression,

$$C = C^1 + C^2,$$ (104)

one obtains C:

$$C = M^3 R + M^5 E\star$$

$$= \frac{G_2\, R}{1 + G_2} + \frac{D\star G_1 G_2}{1 + G_2}\, E\star$$

$$= R_1 + D\star\, G_3 E\star.$$ (105)

Take the pulse transforms of each side of the Equations (101), (103), and (105).

$$E\star = R_2\star - D\star G_3 E\star$$ (106)

$$Y\star = R_2\star + \left(\frac{G_1}{1 + G_2}\right)^\star D\star E\star$$ (107)

$$C\star = \left(\frac{G_2 R}{1 + G_2}\right)^\star + \left(\frac{G_1 G_2}{1 + G_2}\right)^\star D\star E\star$$

$$= R_1\star + G_3\star D\star E\star$$ (108)

Draw the sampled SFG from the pulsed Equations (106) through (108) (Figure 9). Note this is not the same as the sampled SFG that resulted from using the modified SFG method of Section 3. However, the final answers — the desired responses — will be the same, as will be seen in the sequel.

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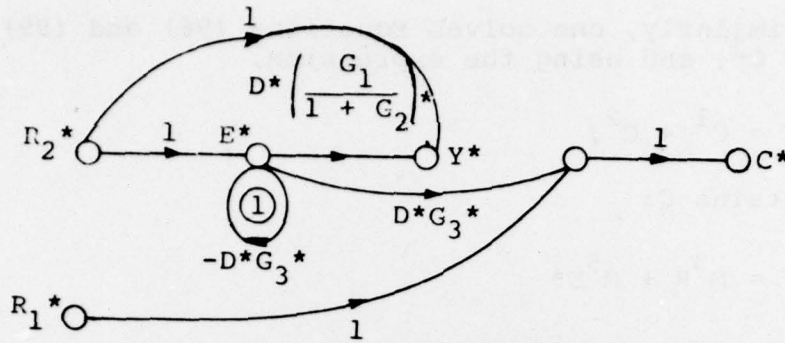
Note: Encircled numbers refer to identification of loops.

Figure 9. Sampled SFG for Example No. 2
(standard SFG method).

STEP NO. 3.

Obtain C^* from the sampled SFG (<u>Figure 9</u>), using Mason's Gain Rule. There are two inputs: $\overline{R_1^*}$ and $\overline{R_2^*}$. They have two forward paths to C^*; $k = 1$ and $k = 2$, respectively. Again using techniques that have been well established in this report:

$$M_1 = 1, \tag{109}$$

$$M_2 = D^*G_3^*, \tag{110}$$

$$K_1 = -D^*G_3^*, \tag{111}$$

$$\Delta = 1 - K_1 = 1 + D^*G_3^*, \tag{112}$$

$$\Delta_1 = \Delta, \tag{113}$$

$$\Delta_1 = 1, \tag{114}$$

$$M^1 \underset{=}{\mathrm{d}}. \ \frac{C^{1*}}{R_1^*} = \frac{M_1 \Delta_1}{\Delta} = 1, \tag{115}$$

$$M^2 \underset{=}{\mathrm{d}}. \ \frac{C^{2*}}{R_2^*} = \frac{M_2 \Delta_2}{\Delta} = \frac{D^*G_3^*}{1 + D^*G_3^*} . \tag{116}$$

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One solves Equations (115) and (116) for C^{1*} and C^{2*}, respectively. Using the expression,

$$C^* = C^{1*} + C^{2*}, \tag{117}$$

one obtains C^*:

$$C^* = M^1 R_1^* + M^2 R_2^*. \tag{118}$$

Equation (118) may be manipulated into the same form as Equations (25) and (45), if desired, by substituting the expression [resulting from Equation (28)].

$$R^* - R_1^* = R_2^* \tag{119}$$

into Equation (118):

$$C^* = R_1^* + \frac{D^* G_3^*}{1 + D^* G_3^*} \ R_2^*$$

$$= \frac{R_1^*(1 + D^* G_3^*) + D^* G_3^*(R^* - R_1^*)}{1 + D^* G_3^*}$$

$$= \frac{R_1^* + D^* G_3^* R^*}{1 + D^* G_3^*} \ . \tag{120}$$

STEP NO. 4. OBTAIN C

Connect the SFG's of <u>Figures 6 and 9</u> to obtain a composite SFG (<u>Figure 10</u>).
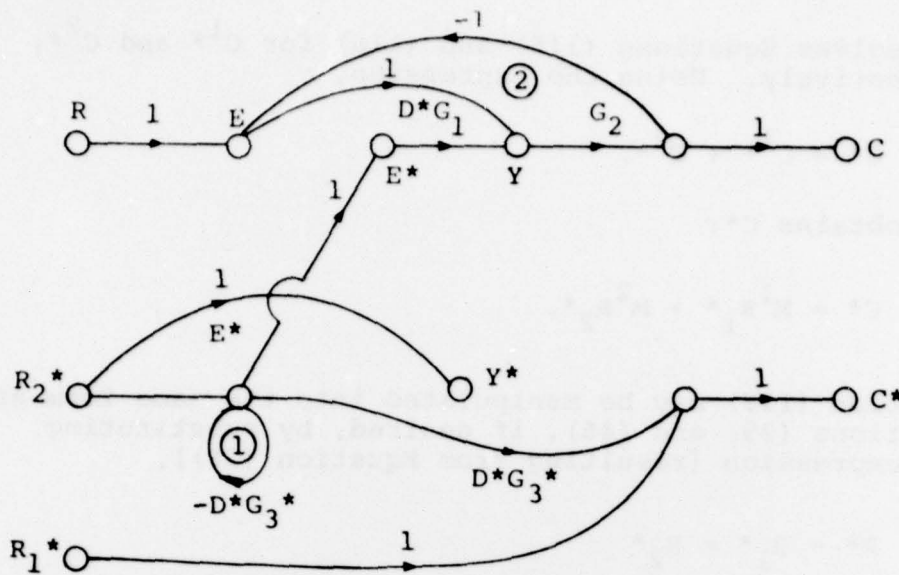
Note: Encircled numbers refer to identification of loops.

Figure 10. Composite SFG for Example No. 2
(Standard SFG method).

Apply Mason's Gain Rule to Figure 10 to obtain C. There are three inputs to the composite system: R, R_1^*, and R_2^*. There is one forward path from R to C (k = 1), no forward paths from R_1^*, and one from R_2^* (k = 2).

$$M_1 = G_2 \tag{121}$$

$$M_2 = D^*G_1G_2 \tag{122}$$

There are now two loops (see Figure 10): $\qquad$ (123)

$$K_1 = -D^*G_3^*,$$

$$K_2 = -G_2,$$

$$\begin{aligned}
\Delta &= 1 - (K_1 + K_2) + K_1K_2 \\
&= 1 - (-D^*G_3^* - G_2) + (-D^*G_3^*)(-G_2) \\
&= 1 + D^*G_3^* + G_2 + G_2G_3^*D^* \\
&= (1 + D^*G_3^*)(1 + G_2),
\end{aligned} \tag{125}$$

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$$\Delta_1 = 1 - K_1 = 1 + D^*G_3^*, \qquad (126)$$

$$\Delta_2 = 1, \qquad (127)$$

$$M^1 \stackrel{\mathrm{d}}{=} \cdot \frac{C^1}{R} = \frac{M_1 \Delta_1}{\Delta} = \frac{G_2(1 + D^*G_3^*)}{(1 + D^*G_3^*)(1 + G_2)} = \frac{G_2}{1 + G_2}, \qquad (128)$$

$$M^2 \stackrel{\mathrm{d}}{=} \cdot \frac{C^2}{R_2^*} = \frac{M_2 \Delta_2}{\Delta}$$

$$= \frac{D^*G_1 G_2}{(1 + D^*G_3^*)(1 + G_2)}. \qquad (129)$$

Using the expression,

$$C = C^1 + C^2, \qquad (130)$$

and solving Equations (128) and (129) for C^1 and C^2, respectively, one may obtain

$$C = \frac{G_2 R}{1 + G_2} + \frac{D^*G_1 G_2 R_2^*}{(1 + D^*G_3^*)(1 + G_2)}$$

$$= R_1 + \frac{G_3 D^* R_2^*}{1 + D^*G_3^*}. \qquad (131)$$

It is noted that the values of C* and C just obtained are the same as those obtained using the SAM and modified SFG techniques.

5. APPLICATION TO Z- AND MODIFIED Z-TRANSFORMS

In the SAM and SFG techniques for obtaining sampled-data outputs of a system, that form has been indicated as C* or C*(s). The z-transform of C*(s) is merely written as C(z). Hence, anywhere an expression C*(s) is found, it may be replaced by C(z) if it is desired to work in the z-domain rather than the s-domain.

If it is desired to find an output expression in modified z-transform, that is denoted by the symbol $C(z,m)$. This form may readily be obtained from the expression for a sampled output, such as C^* or $C^*(s)$, by noting that such outputs appear to be equal to the product of an unstarred quantity and a starred quantity. Let $A(s)$ represent the unstarred quantity, and let $B^*(s)$ represent the starred quantity. Then variable $C^*(s)$ may be written as

$$C^* = C^*(s) = A(s) \ B^*(s). \tag{132}$$

If one recognizes that $A(s)$ or $B^*(s)$ may be equal to unity, Equation (132) will always hold.

The modified z-transform may always be obtained from Equation (132) by performing the following transformation:

$$C(z,m) = A(z,m) \ B(z), \tag{133}$$

where $A(z,m)$ represents the modified z-transform of the quantity $A(s)$, and $B(z)$ represents the ordinary z-transform of the quantity $B(s)$. This technique appears to be an attractive alternative to obtaining modified z-transforms through SFG techniques (which may of course be done).

6. COMPARISON OF METHODS

The Systematic Analysis Method (SAM) can be used to determine the states of a digital system in terms of that system's transfer functions and the inputs to the system. This can also be done by the application of other methods, such as SFG techniques. The usual advertised advantages of the latter, when they are compared to block diagram or algebraic manipulation, is that they are particularly amenable to the analysis of complicated systems.

It has been demonstrated in this report that SAM can handle digital system analysis as capably as can SFG methods. It has the advantage of not requiring the cumbersome Mason's Gain Rule. Hence, it avoids the oft-committed errors associated with SFG analysis, such as overlooked closed loops, nonobvious forward paths, etc. As with SFG's, a block diagram is not needed; the system equations are sufficient. Finally, it has been shown that SAM is easy to implement. It appears to take less lengthy analytical manipulation.

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If the analyst prefers using SFG's to either block diagrams or algebraic manipulation, a modified SFG technique based on SAM techniques is proposed. As such, it is systematic. While the SFG's produced by this technique are usually different from those produced by standard SFG techniques, they yield the same results. Mason's Gain Rule is only applied at one stage of the analysis in the modified SFG technique, as opposed to the standard SFG technique which requires several applications of Mason's Gain Rule.

7. CONCLUSIONS

An alternative to the Signal Flow Graph technique has been presented and compared to a standard and a modified SFG. The alternative, termed Systematic Analytical Method or SAM, is claimed herein to be simpler and more straightforward to implement than the SFG methods. Not only does it appear to be quicker to use in system analysis, but it obviates the cumbersome use of Mason's Gain Rule.

For the analyst who desires to use SFG techniques, a modified SFG technique is presented. It is systematic and reduces the number of required applications of Mason's Gain Rule.

APPENDIX A - REVIEW OF MASON'S GAIN RULE

M: The gain (transfer function) between two nodes on a Signal Flow Graph (SFG).

k: The number of forward paths leading from all system inputs to a particular selected output.

M_k: The gain along the kth forward path.

$\Delta_1 - \sum$ (<u>all</u> individual loop gains)

$+ \sum$ (gain products of all possible combinations of two nontouching loops)

$- \sum$ (gain products of all possible combinations of three nontouching loops)

$+ \ldots$

Δ_k = Value of Δ for that part of the graph not touching the kth forward path.

$$M = \sum_k \frac{M_k \Delta_k}{\Delta} \ .$$

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